Newton polygons II

Wednesday, January 6, 2021 9:35 AM

Recall Lazard's thm:

Let K be a non-arch field

$f \in \mathcal{O}_K[[T]]$ a series

Def $\mathrm{Newt}(f) :=$ decreasing lower convex hull of $\{(i, v(a_i))\}$, $i = 0, 1, \dots, \infty$

$$\left(f = \sum_{i \geq 0} a_i T^i, \quad a_i \in \mathcal{O}_K \right)$$

Thm (Lazard)

If $\lambda \neq 0$ is a slope of $\mathrm{Newt}(f)$, $\Rightarrow \exists\, \alpha \in \hat{\bar{K}}$ w/ $f(\alpha) = 0$ & $v(\alpha) = -\lambda$

Goal: "prove" sthg similar for $A_{\inf}$

Recall Setup

$E/\mathbb{Q}_p$ finite, π uniformizer, $\mathcal{O}_E/\pi \simeq \mathbb{F}_q$

$F/\mathbb{F}_q$ alg. closed, complete non-arch extension

$v : F \to \mathbb{R} \cup \{\infty\}$

(allowed to think $E = \mathbb{Q}_p$, $F = \overline{\mathbb{F}_p((T))}$)

$$A_{\inf} := W_{\mathcal{O}_E}(\mathcal{O}_F) \quad (= W(\mathcal{O}_F))$$

As $\mathcal{O}_F$ is perfect $\mathbb{F}_p$-alg, $\exists$ unique Teichmüller expansion:

$$[\ ] : \mathcal{O}_F \to W(\mathcal{O}_F) \text{ mult.}$$

& $A_{\inf} = \left\{ \sum [a_i] p^i, \ a_i \in \mathcal{O}_F \right\}$

Guess $f \in A_{\inf}$, decreasing lower

... (f) = convex hull of

$\mathrm{Newt}(f) =$ decreasing convex hull of $\{(i, v(a_i))\}_{i \in \mathbb{N}}$

Last time:

$\mathrm{Newt}(fg) = \mathrm{Newt}(f) \star \mathrm{Newt}(g)$

$=$

Notation

$|Y|_{[0,\infty)} := \{ I \subseteq A_{\inf} \mid I$ gen by a distinguished elt $\}$

$= \mathrm{Prim}_{\underline{\Delta}} \mid A_{\inf}$ ✓

"0" $\longleftrightarrow (p)$

$|Y| = |Y|_{[0,\infty)} \setminus \{(p)\}$

$|Y| \longleftrightarrow \{ (C, \iota) \mid C/\mathbb{Q}_p$ non-arch ext. $\iota: \mathcal{O}_C^{\flat} \xrightarrow{\sim} \mathcal{O}_F$ $\}$

$|Y| \longleftarrow$ { | $\iota: \mathcal{O}_C^{\flat} \xrightarrow{\sim} \mathcal{O}_F$ }

"char 0 untilts of $\mathcal{O}_F$"

$y \in |Y|$

- $\mathfrak{p}_y \subseteq A_{inf}$ is the principal ideal
- $(\xi_y) = \mathfrak{p}_y$ is generator
- $C_y := (A_{inf}/\mathfrak{p}_y)[\frac{1}{p}]$
- $\Theta_y : A_{inf} \longrightarrow C_y$
- $v_y : C_y \longrightarrow \mathbb{R} \cup \{\infty\}$

  $v_y(\Theta_y([a])) = v(a)$

  $(a \in \mathcal{O}_F)$

- $f \in A_{inf}$, $f(y) := \Theta_y(f) \in C_y$

Thm

Thm

Let $f \in A_{\inf}$, $\lambda \neq 0$ slope of $\mathrm{Newt}(f)$. Then $\exists\, \alpha \in \mathcal{O}_F$, w/ $\mathrm{val}(\alpha) = -\lambda$, s.t.

$$f = (p - [\alpha])\, g, \quad g \in A_{\inf}$$

Warning: α is not unique

One key idea

$\exists$ notion of distance of $|Y|$

Def If $y_1, y_2 \in |Y|$

$$d(y_1, y_2) := v_{y_1}(\theta_{y_1}(\xi_{y_2}))$$

$A_{\inf}$

C_{y_1}

$\mathbb{R} \cup \{\infty\}$

$$d(y,0) := v(p(y))$$

image of p in C_y

Claim:

① d is an ultrametric valuation

② "unit circle" is complete w/r/t d